

Positive integers: counterexample to W.M. Schmidt's conjecture

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Abstract.

We show that there exist real numbers α_1, α_2 linearly independent over \mathbb{Z} together with 1 such that for every non-zero integer vector (m_1, m_2) with $m_1 \geq 0$ and $m_2 \geq 0$ one has $\|m_1\alpha_1 + m_2\alpha_2\| \geq 2^{-300}(\max(m_1, m_2))^{-\sigma}$ with $\sigma = 1.94696^+$.

1 Introduction

Let $\|\xi\|$ denotes the distance from real ξ to the nearest integer. Let $\phi = \frac{1+\sqrt{5}}{2}$. In [1] W.M. Schmidt proved the following result.

Theorem A. (W.M. Schmidt) *Let real numbers α_1, α_2 be linearly independent over \mathbb{Z} together with 1. Then there exists a sequence of integer two-dimensional vectors $(x_1(i), x_2(i))$ such that*

1. $x_1(i), x_2(i) > 0$;
2. $\|\alpha_1 x_1(i) + \alpha_2 x_2(i)\| \cdot (\max\{x_1(i), x_2(i)\})^\phi \rightarrow 0$ as $i \rightarrow +\infty$.

W.M. Schmidt posed a conjecture that the exponent ϕ here may be replaced by $2 - \varepsilon$ with arbitrary positive ε (see [2]). In this paper we show this conjecture to be false.

Let $\sigma = 1.94696^+$ be the largest real root of the equation

$$x^4 - 2x^2 - 4x + 1 = 0. \quad (1)$$

Theorem 1. *There exist real numbers α_1, α_2 such that they are linearly independent over \mathbb{Z} together with 1 and for every integer vector $(m_1, m_2) \in \mathbb{Z}^2$ with $m_1, m_2 \geq 0$ and $\max(m_1, m_2) \geq 2^{200}$ one has*

$$\|m_1\alpha_1 + m_2\alpha_2\| \geq \frac{1}{2^{300}(\max(m_1, m_2))^\sigma}.$$

We would like to formulate a related result from our paper [3]. For a real $\gamma \geq 2$ we define a function

$$g(\gamma) = \phi + \frac{2\phi - 2}{\phi^2\gamma - 2}.$$

One can see that $g(\gamma)$ is a strictly decreasing function and

$$g(2) = 2, \quad \lim_{\gamma \rightarrow +\infty} g(\gamma) = \phi.$$

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For positive Γ define

$$C(\Gamma) = 2^{18} \Gamma^{\frac{\phi - \phi^2}{\phi^2 \gamma - 2}}.$$

In [3] the following statement was proved.

Theorem B. *Suppose that real numbers α_1, α_2 satisfy the following Diophantine condition. For some $\Gamma \in (0, 1)$ and $\gamma \geq 2$ the inequality*

$$\|\alpha_1 m_1 + \alpha_2 m_2\| \geq \frac{\Gamma}{(\max\{|m_1|, |m_2|\})^\gamma} \quad (2)$$

holds for all integer vectors $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Then there exists an infinite sequence of integer two-dimensional vectors $(x_1(i), x_2(i))$ such that

1. $x_1(i), x_2(i) > 0$;
2. $\|\alpha_1 x_1(i) + \alpha_2 x_2(i)\| \cdot (\max\{x_1(i), x_2(i)\})^{g(\gamma)} \leq C(\Gamma)$ for all i .

Of course constants 2^{200} , 2^{300} and 2^{18} in Theorem 1 and in the definition of $C(\Gamma)$ may be reduced.

2 The construction

We shall deal with the Euclidean norm for simplicity reason. So we use $|\cdot|$ for the Euclidean norm of two- or three-dimensional vectors. By $\angle(\mathbf{u}, \mathbf{v})$ we denote the angle between vectors \mathbf{u}, \mathbf{v} .

Define

$$\tau = \frac{1 + \sigma^2}{2\sigma} = 1.23029^+. \quad (3)$$

Note that

$$\sigma\tau - 1 > \tau. \quad (4)$$

Put

$$\omega = \tau + 1. \quad (5)$$

Fundamental Lemma. *There exist real numbers $\alpha_1, \alpha_2 \in \mathbb{R}$ linearly independent together with 1 over \mathbb{Z} and such that there exists a sequence of integer vectors*

$$\mathbf{m}_0 = (1, 1, -1), \quad \mathbf{m}_\nu = (m_{0,\nu}, m_{1,\nu}, m_{2,\nu}) \in \mathbb{Z}^3, \nu = 1, 2, 3, \dots$$

satisfying the following conditions (i) – (v).

(i) *For any $\nu \geq 1$ the triple $\mathbf{m}_{\nu-1}, \mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ consists of linearly independent vectors, and each two-dimensional sublattice*

$$\mathcal{L}_\nu = \langle \mathbf{m}_\nu, \mathbf{m}_{\nu+1} \rangle_{\mathbb{Z}}$$

is complete, that is

$$\mathbb{Z}^3 \cap \text{span } \mathcal{L}_\nu = \mathcal{L}_\nu, \quad \nu = 0, 1, 2, 3, \dots$$

(ii) *Define*

$$\zeta_\nu = m_{0,\nu} + m_{1,\nu}\alpha_1 + m_{2,\nu}\alpha_2, \quad M_\nu = |\overline{\mathbf{m}}_\nu|.$$

For every $\nu \geq 0$ one has

$$\frac{1}{2^5 M_{\nu+1}^\omega} \leq \zeta_\nu \leq \frac{1}{M_{\nu+1}^\omega}. \quad (6)$$

(iii) $M_1 \leq 2^{100}$ and for every $\nu \geq 1$ one has

$$2^{10}M_\nu \leq M_{\nu+1} \quad (7)$$

and

$$H_\nu \leq M_{\nu+1} \leq 2H_\nu, \quad H_\nu = \frac{M_\nu^{\sigma\tau-1}}{2^9}. \quad (8)$$

(iv) For every $\nu \geq 0$ one has $m_{1,\nu} \cdot m_{2,\nu} < 0$; moreover for the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\overline{\mathbf{m}}_\nu = (m_{1,\nu}, m_{2,\nu}) \in \mathbb{Z}^2$$

one has

$$\text{angle}(\overline{\mathbf{m}}_\nu, \pm \mathbf{e}_j) \geq \frac{1}{4}, \quad j = 1, 2. \quad (9)$$

(v) For every $\nu \geq 0$ for vectors

$$\overline{\mathbf{m}}_\nu = (m_{1,\nu}, m_{2,\nu}), \quad \overline{\mathbf{m}}_{\nu+1} = (m_{1,\nu+1}, m_{2,\nu+1})$$

one has

$$\text{angle}(\overline{\mathbf{m}}_\nu, \pm \overline{\mathbf{m}}_{\nu+1}) \geq \frac{1}{4}.$$

We give a sketched proof of Fundamental Lemma in Section 6. It use standard argument related to an inductive construction of special singular (in the sense of A. Khintchine) vectors. Inequality (4) is of major importance. Many different properties of singular vectors are discussed in our recent survey [4].

For every ν we define two-dimensional lattice $\Lambda_\nu = \langle \overline{\mathbf{m}}_\nu, \overline{\mathbf{m}}_{\nu+1} \rangle_{\mathbb{Z}} \subset \mathbb{Z}^2$. Let D_ν be the fundamental volume of the lattice Λ_ν . Obviously

$$D_\nu \leq M_\nu M_{\nu+1}. \quad (10)$$

From the condition (v) one has

$$D_\nu \geq \frac{M_\nu M_{\nu+1}}{2^5}. \quad (11)$$

In the sequel we use the following notation. For an integer vector $\mathbf{m} = (m_0, m_1, m_2) \in \mathbb{Z}^3$ we define

$$\zeta = \zeta(\mathbf{m}) = m_0 + m_1\alpha_1 + m_2\alpha_2, \quad \overline{\mathbf{m}} = \overline{\mathbf{m}}(\mathbf{m}) = (m_1, m_2) \in \mathbb{Z}^2$$

and

$$M = M(\mathbf{m}) = |\overline{\mathbf{m}}|.$$

In Sections 3,4,5 below we suppose that α_1, α_2 are the numbers from Fundamental Lemma.

3 Linearly independent vectors

We prove a lemma concerning a lower bound for the value of $|\zeta(\mathbf{m})|$ in the case when the vector $\mathbf{m} \in \mathbb{Z}^3$ is linearly independent of vectors $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$.

Consider the segment

$$\mathcal{I}_\nu = [(4M_\nu M_{\nu+1})^{1/\sigma}, M_{\nu+1}^\tau/8] \quad (12)$$

(inequalities (4) and (7) show that the left endpoint of the segment is less than the right endpoint indeed).

Lemma 1. *Suppose that a vector $\mathbf{m} \in \mathbb{Z}^3$ is linearly independent of vectors $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ and*

$$M \in \mathcal{I}_\nu. \quad (13)$$

Then

$$|\zeta(\mathbf{m})| \geq M^{-\sigma}.$$

Proof.

Consider the determinant

$$\Delta = \begin{vmatrix} m_0 & m_1 & m_2 \\ m_{0,\nu} & m_{1,\nu} & m_{2,\nu} \\ m_{0,\nu+1} & m_{1,\nu+1} & m_{2,\nu+1} \end{vmatrix} = \begin{vmatrix} \zeta(\mathbf{m}) & m_1 & m_2 \\ \zeta_\nu & m_{1,\nu} & m_{2,\nu} \\ \zeta_{\nu+1} & m_{1,\nu+1} & m_{2,\nu+1} \end{vmatrix}.$$

We see from (6, 5) that

$$1 \leq |\Delta| \leq 2|\zeta(\mathbf{m})|M_\nu M_{\nu+1} + 4MM_{\nu+1}^{-\tau}.$$

From the inequality $M \leq M_{\nu+1}^\tau/8$ which follows from (13) we see that $4MM_{\nu+1}^{-\tau} \leq 1/2$. That is why $|\zeta(\mathbf{m})|M_\nu M_{\nu+1} \geq 1/4$. Now we take into account the lower bound for M from (13) and the lemma follows. \square

4 Vectors dependent with $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$

Condition (i) means that each integer vector $\mathbf{m} \in \mathbb{Z}^3$ which is linearly dependent together with $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ can be written in a form

$$\mathbf{m} = \lambda \mathbf{m}_\nu + \mu \mathbf{m}_{\nu+1}$$

with integer λ and μ . So if $\mathbf{m} \in \mathbb{Z}^3$ is linearly dependent together with $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ then for “cutten” vectors we have the equality

$$\overline{\mathbf{m}} = \lambda \overline{\mathbf{m}}_\nu + \mu \overline{\mathbf{m}}_{\nu+1} \quad (14)$$

with integer λ and μ .

Lemma 2. *Suppose that the vector $\mathbf{m} = (m_0, m_1, m_2) \in \mathbb{Z}^3$ satisfy the condition $m_1, m_2 \geq 0$. Suppose that vectors $\mathbf{m}, \mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ are linearly dependent for some ν . Then*

$$|\zeta(\mathbf{m})| \geq 2^{-300} M^{-\sigma}.$$

Proof.

We can split two-dimensional lattice Λ_ν into a countable union of one-dimensional lattices $\Lambda_{\nu,\mu}$ in the following way:

$$\Lambda_\nu = \bigsqcup_{\mu \in \mathbb{Z}} \Lambda_{\nu,\mu}, \quad \Lambda_{\nu,\mu} = \{\mathbf{z} = (z_1, z_2) \in \Lambda_\nu : \mathbf{z} = \lambda \overline{\mathbf{m}}_\nu + \mu \overline{\mathbf{m}}_{\nu+1}, \lambda \in \mathbb{Z}\}.$$

By the condition (iv) there is no non-zero points $(z_1, z_2) \in \Lambda_{\nu,0}$ satisfying $z_1 \cdot z_2 \geq 0$.

Suppose that $\mu \neq 0$. As the fundamental volume of Λ_ν is equal to D_ν we see that the Euclidean distance between any two neighbouring lines aff $\Lambda_{\nu,\mu}$ and aff $\Lambda_{\nu,\mu+1}$ is equal to $D_\nu / \sqrt{m_{1,\nu}^2 + m_{2,\nu}^2}$. That is why the conditions

$$(m_1, m_2) \in \Lambda_{\nu,\mu}, \quad m_1, m_2 \geq 0 \quad (15)$$

imply

$$\max(m_1, m_2) \geq \frac{|\mu|D_\nu}{2M_\nu} \geq \frac{|\mu|M_{\nu+1}}{2^8} \quad (16)$$

(in the last inequality we use (11)).

From the other hand conditions (15) together with (9) from (iv) lead to the inequality

$$|\lambda| \geq \frac{|\mu|M_{\nu+1}}{2^8 M_\mu}$$

for the coefficient λ from (14). So (we apply lower bound from (ii) for ζ_ν and upper bound from (ii) for $\zeta_{\nu+1}$) we see that

$$|\zeta(\mathbf{m})| = |\lambda\zeta_\nu + \mu\zeta_{\nu+1}| \geq |\mu| \left(\frac{M_{\nu+1}}{2^8 M_\nu} \zeta_\nu - \zeta_{\nu+1} \right) \geq |\mu| \left(\frac{1}{2^{13} M_\nu M_{\nu+1}^\tau} - \frac{1}{M_{\nu+2}^{\tau+1}} \right).$$

We apply lower bound (7) from (iii) to get

$$|\zeta(\mathbf{m})| \geq \frac{|\mu|}{2^{14} M_\nu M_{\nu+1}^\tau}.$$

Now lower bound (8) from (iii) gives

$$|\zeta(\mathbf{m})| \geq 2^{-14 - \frac{100}{\sigma\tau-1}} |\mu| M_{\nu+1}^{\tau + \frac{1}{\sigma\tau-1}} = 2^{-14 - \frac{100}{\sigma\tau-1}} |\mu| M_{\nu+1}^{-\sigma} > 2^{-200} |\mu| M_{\nu+1}^{-\sigma} \quad (17)$$

(here we use the definition of σ as a root of (1) and (3) to see that $\tau + \frac{1}{\sigma\tau-1} = \sigma$). Now we combine (16,17) and (8) from the condition (iii) to get

$$|\zeta(\mathbf{m})| \geq \frac{|\mu|^{1+\sigma}}{2^{300} M^\sigma} > \frac{1}{2^{300} M^\sigma}.$$

Lemma 2 is proved. \square

5 Proof of Theorem 1

We take α_1, α_2 from Fundamental Lemma. Consider an integer vector $\mathbf{m} = (m_0, m_1, m_2)$ with $m_1, m_2 \geq 0$. We may suppose that $|\zeta(\mathbf{m})| = ||m_1\alpha_1 + m_2\alpha_2||$. If for some ν vectors

$$\mathbf{m}, \mathbf{m}_\nu, \mathbf{m}_{\nu+1} \quad (18)$$

are linearly dependent then application of Lemma 2 proves Theorem 1. So we may suppose that all triples (18) consist of linearly independent vectors for every $\nu \geq 0$. Now to prove Theorem 1 we may use Lemma 1. It is enough to show that

$$\bigcup_{\nu \geq 0} \mathcal{I}_\nu \supset [2^{200}, +\infty)$$

(segments \mathcal{I}_ν are defined in (12)). But this follows from the condition $M_1 \leq 2^{100}$ and the inequality

$$(4M_\nu M_{\nu+1})^{1/\sigma} \leq M_\nu^\tau / 8.$$

The last inequality is a corollary of the right inequality from (8). \square

6 Fundamental Lemma: sketch of a proof

Let $\mathbf{m} \in \mathbb{Z}^3$ be an integer vector. The formulation of Fundamental Lemma deals with the values of $M = M(\mathbf{m}) = |\overline{\mathbf{m}}|$. To describe the ideas of the proof it is much more convenient to consider the Euclidean norm $M = |\mathbf{m}|$ of the vector \mathbf{m} itself than the Euclidean norm $M = |\overline{\mathbf{m}}|$ of the “cutten” vector $\overline{\mathbf{m}} \in \mathbb{Z}^2$. Of course values of M and M are of the same order for all integer vectors \mathbf{m} under consideration. We may assume that $M \leq M \leq 2M$.

Let

$$\mathfrak{S} = \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$$

be the unit sphere. We construct a sequence of nested closed sets $\mathcal{B}_\nu \subset \mathfrak{S}$ by induction. Their unique common point $\mathbf{x}^* = (x_0^*, x_1^*, x_2^*) \in \bigcap_\nu \mathcal{B}_\nu$ will define real numbers $\alpha_1 = x_1^*/x_0^*, \alpha_2 = x_2^*/x_0^*$ which satisfy the conclusion of Fundamental Lemma.

The base of inductive process is trivial.

To proceed the inductive step we suppose that the following objects are already constructed:

- 1) primitive integer vectors $\mathbf{m}_j = (m_{0,j}, m_{1,j}, m_{2,j})$, $0 \leq j \leq \nu$ with $M_j = |\mathbf{m}_j|$; we suppose that these vectors satisfy conditions (iv), (v);
- 2) vectors $\xi_j = (\xi_{0,j}, \xi_{1,j}, \xi_{2,j}) \in \mathfrak{S}$ such that $\xi_j \perp \mathbf{m}_j$, $\xi_j \perp \mathbf{m}_{j+1}$, $0 \leq j \leq \nu - 1$, and one-dimensional linear subspaces $\Xi_j = \text{span } \xi_j$, $0 \leq j \leq \nu - 1$;
- 3) two-dimensional linear subspaces

$$\ell_j^0 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : m_{0,j}x_0 + m_{1,j}x_1 + m_{2,j}x_2 = 0\}, \quad 0 \leq j \leq \nu - 1,$$

and two-dimensional affine subspaces

$$\ell_j^1 = \left\{ (x_0, x_1, x_2) \in \mathbb{R}^3 : m_{0,j}x_0 + m_{1,j}x_1 + m_{2,j}x_2 = \frac{1}{2M_{j+1}^\omega} \right\}, \quad 0 \leq j \leq \nu - 1;$$

- 4) cylinders

$$\mathcal{C}_j = \left\{ \mathbf{x} \in \mathbb{R}^3 : \text{dist}(\mathbf{x}, \Xi_j) \leq \frac{1}{M_{j+1}^\omega M_j} \right\}, \quad 0 \leq j \leq \nu - 1$$

(here $\text{dist}(\cdot, \cdot)$ denotes the Euclidean distance between sets) and closed sets

$$\mathcal{G}_j = \left\{ \mathbf{x} \in \mathfrak{S} \cap \mathcal{C}_j : m_{0,j}x_0 + m_{1,j}x_1 + m_{2,j}x_2 \geq \frac{1}{2M_{j+1}^\omega} \right\} \subset \mathfrak{S}, \quad 0 \leq j \leq \nu - 1$$

(so a part of the boundary of \mathcal{G}_j belongs to ℓ_j^1);

- 5) two-dimensional complete sublattices $\mathcal{L}_j = \langle \mathbf{m}_j, \mathbf{m}_{j+1} \rangle_{\mathbb{Z}}$, $j = 0, \dots, \nu - 1$ with fundamental volumes d_j satisfying inequalities

$$\frac{M_j M_{j+1}}{2^5} \leq d_j \leq M_j M_{j+1} \tag{19}$$

(here the right inequality is trivial, the left one means that the angle between vectors $\mathbf{m}_{j-1}, \mathbf{m}_j$ is bounded from below);

- 6) we suppose that the vector \mathbf{m}_ν is defined, so we can consider linear subspace

$$\ell_\nu^0 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : m_{0,\nu}x_0 + m_{1,\nu}x_1 + m_{2,\nu}x_2 = 0\};$$

we suppose that linear subspaces ℓ_j^0 for every j from the range $1 \leq j \leq \nu$ satisfy the condition

$$\ell_j^0 \cap \mathcal{G}_{j-1} \neq \emptyset, \quad 1 \leq j \leq \nu;$$

moreover we suppose that for any j from the range $1 \leq j \leq \nu$ there is a point $\eta_j = (\eta_{0,j}, \eta_{1,j}, \eta_{2,j}) \in \ell_j^0 \cap \mathcal{G}_{j-1}$ such that the set

$$\mathcal{B}_j = \left\{ \mathbf{x} \in \mathfrak{S} : |\mathbf{x} - \eta_j| \leq \frac{1}{2^6 M_j^\omega M_{j-1}} \right\}$$

satisfy the condition

$$\mathcal{B}_j \subset \mathcal{G}_{j-1} \subset \mathcal{B}_{j-1}. \quad (20)$$

Here we should note that $\mathcal{B}_0 \supset \mathcal{B}_1 \supset \dots \supset \mathcal{B}_{\nu-1} \supset \mathcal{B}_\nu$.

We suppose that vectors \mathbf{m}_j , $1 \leq j \leq \nu$ and every couple α_1, α_2 of the form $\alpha_1 = x_1/x_0, \alpha_2 = x_2/x_0$, $\mathbf{x} = (x_0, x_1, x_2) \in B_{\nu-1}$ satisfy all the conditions (i) – (v) of Fundamental Lemma which are defined up to the $(\nu - 1)$ -th step.

Our task is to define an integer vector $\mathbf{m}_{\nu+1}$ and all related objects of the ν -th step.

Consider $\mathbf{n} = (n_0, n_1, n_2) \in \mathbb{Z}^3$ such that the triple $\mathbf{n}, \mathbf{m}_{\nu-1}, \mathbf{m}_\nu$ form a basis of \mathbb{Z}^3 . Such vector does exist as the lattice $\mathcal{L}_{\nu-1}$ is complete. We may suppose that

$$\max(|n_1|, |n_2|) \leq M_\nu. \quad (21)$$

We consider two-dimensional lattices

$$\mathcal{L}_{\nu-1, \mu} = \{ \mathbf{z} = \lambda_1 \mathbf{m}_{\nu-1} + \lambda_2 \mathbf{m}_\nu + \mu \mathbf{n}, \quad \lambda_1, \lambda_2 \in \mathbb{Z} \}.$$

Note that

$$\mathbb{Z}^3 = \bigsqcup_{\mu \in \mathbb{Z}} \mathcal{L}_{\nu-1, \mu}.$$

In fact $\mathbf{n} \in \mathcal{L}_{\nu-1, 1}$. The Euclidean distance between the neighbouring affine subspaces $\text{aff } \mathcal{L}_{\nu-1, \mu}$ and $\text{aff } \mathcal{L}_{\nu-1, \mu+1}$ is equal to $d_{\nu-1}^{-1}$. Put

$$\mu_* = d_{\nu-1} H_\nu M_\nu^{-\omega} M_{\nu-1}^{-1}$$

In fact μ_0 is of the size

$$\mu_* \asymp M_\nu^t, \quad t = \sigma\tau - \omega > 0$$

(here the last inequality follows from (4)).

Now here we define two-dimensional linear subspace $\ell_\nu^* \subset \mathbb{R}^3$ and a point $\mathbf{w}_\nu \in \text{aff } \mathcal{L}_{\nu-1, \mu_*}$ by the following way. Consider the unique one-dimensional affine subspace $\pi \subset \mathbb{R}^3$ such that

- 1) $\eta_\nu \in \pi$,
- 2) π is parallel to $\ell_{\nu-1}^0$,
- 3) the intersection $\pi \cap \mathfrak{S}$ consists of just one point η_ν .

We define ℓ_ν^* as follows: $\ell_\nu^* = \text{span } \pi$. Let $\mathbf{w}_\nu \in \text{aff } \mathcal{L}_{\nu-1, \mu_*}$ be the unique point such that $\mathbf{w}_\nu \perp \ell_\nu^*$. Now we define the disk

$$\mathcal{D}_\nu = \left\{ \mathbf{w} \in \text{aff } \mathcal{L}_{\nu-1, \mu_*} : |\mathbf{w} - \mathbf{w}_\nu| \leq \frac{H_\nu}{2^{10}} \right\}.$$

Easy calculation shows that

$$\mathbf{w} \in \mathcal{D}_\nu \implies H_\nu \leq |\overline{\mathbf{w}}| \leq 2H_\nu.$$

For $\mathbf{w} = (w_0, w_1, w_2)$ we consider two-dimensional linear subspace

$$\ell[\mathbf{w}] = \{ \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : w_0 x_0 + w_1 x_1 + w_2 x_2 = 0 \}.$$

Consider a smaller ball $\mathcal{B}'_\nu \subset \mathcal{B}_\nu$ with the same center η_ν and radius $\frac{1}{2^7 M_\nu^{-\omega} M_{\nu-1}}$. Easy calculation shows that

$$\mathbf{w} \in \mathcal{D}_\nu \implies \ell[\mathbf{w}] \cap \ell_\nu^0 \cap \mathcal{B}'_\nu \neq \emptyset.$$

Now if we take an integer vector $\mathbf{m}_{\nu+1} \in \mathcal{D}_\nu$ the conditions (ii), (iii) are satisfied for ν -th step.

Vector ξ_ν , subspaces $\ell_\nu^1, \ell_{\nu+1}^0$ and the set \mathcal{G}_ν are defined automatically. We can easily take $\eta_{\nu+1}$ with all necessary properties, in particular we construct $\mathcal{B}_{\nu+1}$ (the second embedding in (20) with $j = \nu + 1$ follows from the largeness of the value of $M_{\nu+1}$, the first one can be ensured as the angle between vectors $\mathbf{m}_{\nu+1}, \mathbf{m}_\nu$ is almost the same as the angle between vectors $\mathbf{m}_\nu, \mathbf{m}_{\nu-1}$).

Now we must explain how to ensure the condition (i). and (iv), (v).

To get (i) we should note that a vector $\mathbf{n} = (n_0, n_1, n_2) \in \mathcal{L}_{\nu-1,1}$ which completes the pair $\mathbf{m}_{\nu-1}, \mathbf{m}_\nu$ to a basis of \mathbb{Z}^3 may be found in any box of the form

$$A_k \leq n_k \leq A_k + M_\nu, \quad k = 1, 2$$

(this fact follows from (21). For each \mathbf{n} the vector

$$\mathbf{m} = \mu_* \mathbf{n} + \mathbf{m}_{\nu-1} \in \mathcal{L}_{\nu-1, \mu_*}$$

together with \mathbf{m}_ν generates a complete lattice $\langle \mathbf{m}_\nu, \mathbf{m} \rangle_{\mathbb{Z}}$. Note that $M_\nu \mu_* \asymp H_\nu \cdot \frac{d_{\nu-1}}{M_\nu^\omega M_{\nu-1}} = o(H_\nu)$ (we use the upper bound from (19)). So the set of all vectors \mathbf{m} constructed is “dense” in the range $H_\nu \leq |\mathbf{m}| \leq 2H_\nu$. So one can find such a vector with $\mathbf{m} \in \mathcal{D}_\nu$. That is why we can easily satisfy the condition (i) for the ν -th step.

Similarly, as we have many points $\mathbf{m} \in \mathcal{D}_\nu$ satisfying (i) we can take $\mathbf{m}_{\nu+1}$ close enough to \mathbf{w}_ν to satisfy (iv), (v).

As we have certain choice for the vector \mathbf{m}_ν at each step of the inductive construction we can get (α_1, α_2) satisfying linearly independence condition. So the point (α_1, α_2) constructed satisfies all the conditions of Fundamental Lemma. The inductive procedure is described.

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